Control Systems I

State Observers

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State Estimation

Problem: Our controller is a function of the state, but we don't know the state.

Goal: Estimate the state from input and output measurements.

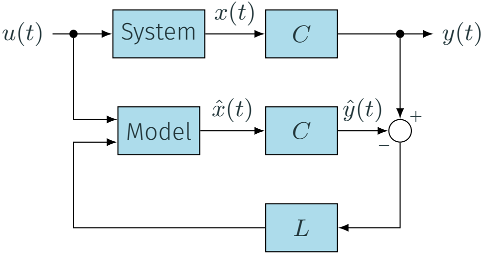
If $\hat{x}$ is our estimate of the state and we simulate the dynamics

$$\dot{\hat{x}} = A\hat{x} + Bu$$

then $\hat{x} \approx x$. Noise and model error will always cause the estimate to diverge.

Solution: Feedback

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$



Outline

Observers

Reference Input

Integral Control

Observers

State Estimation Error Dynamic

Dynamics representing error between the true state and the estimated state is

$$
\begin{aligned}
\dot{e} &= \dot{x} - \dot{\hat{x}} \\
&= Ax + Bu - A\hat{x} - Bu - L(y - C\hat{x}) \\
&= A(x - \hat{x}) - L(Cx - C\hat{x}) \\
&= (A - LC)e
\end{aligned}
$$

Idea: Choose L so that the error system is stable and $\hat{x} \to x$

How: Use pole placement exactly as in the control case

Observer Canonical Form

$$\dot{x} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Error dynamics are

$$A - LC = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -a_1 - L_1 & 1 & 0 \\ -a_2 - L_2 & 0 & 1 \\ -a_3 - L_3 & 0 & 0 \end{bmatrix}$$

with the simple characteristic equation

$$\det(sI - A + LC) = s^3 + (a_1 + L_1)s^2 + (a_2 + L_2)s + (a_3 + L_3) = 0$$

Observer poles can be placed easily if system can be put in observer canonical form

Duality

Pole placement in observer canonical form

$$A - LC = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -a_1 - L_1 & 1 & 0 \\ -a_2 - L_2 & 0 & 1 \\ -a_3 - L_3 & 0 & 0 \end{bmatrix}$$

$$\det(sI - A + LC) = s^3 + (a_1 + L_1)s^2 + (a_2 + L_2)s + (a_3 + L_3)$$

Pole placement in control canonical form

$$A - BK = \begin{bmatrix} -a_1 - K_1 & -a_2 - K_2 & -a_3 - K_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det(sI - A + KB) = s^n + (a_1 + K_1)s^{n-1} + (a_2 + K_2)s^{n-2} + \cdots + (a_3 + K_3)$$

Observer pole placement is identical to controller pole placement if we replace (A, B) with $(A^\intercal, C^\intercal)$

Conversion to Observer Canonical Form

$$G(s) = \frac{7s^2 + 12s + 3}{s^3 + 2s^2 + 5s + 2} = \frac{Y}{U}$$

Divide by s^3 and solve for Y

$$Y = \underbrace{s^{-1}(7U - 2Y) + s^{-2}(12U - 5Y) + s^{-3}(3U - 2Y)}_{X_1}$$

$$sX_1 = 7U - 2Y + \underbrace{s^{-1}(12U - 5Y) + s^{-2}(3U - 2Y)}_{X_2}$$

$$sX_2 = 12U - 5Y + \underbrace{s^{-1}(3U - 2Y)}_{X_3}$$

$$sX_3 = 3U - 2Y$$

$$G(s) = \frac{7s^2 + 12s + 3}{s^3 + 2s^2 + 5s + 2} = \frac{Y}{U}$$

Take inverse Laplace transform

$$\begin{aligned} y &= x_1 & \\ \dot{x}_1 &= 7u + 2y + x_2 & &= 7u + 2x_1 + x_2 \\ \dot{X}_2 &= 12u - 5y + x_3 & &= 12u - 5x_1 + x_3 \\ \dot{X}_3 &= 3u - 2y & &= 3u - 2x_1 \end{aligned}$$

Putting this together, we get observer canonical form

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 \\ -5 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 7 \\ 12 \\ 3 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Ackermann's Estimator Formula

Goal Choose observer gain L for the system (A, C) so that the closed-loop system $\dot{e} = (A - LC)e$ has the characteristic equation $\alpha_e(s)$

$$L = \alpha_e(A)\mathcal{O}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

where $\alpha_e(A)$ is the desired characteristic equation evaluated at the matrix A

$$\alpha_e(A) = A^n + \beta_1 A^{n-1} + \beta_2 A^{n-2} + \cdots + \beta_n$$

$\mathcal{O}$ is the ***observability matrix***, which plays the same role as the controllability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Observability

Consider the system

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

which has the transfer function $G(s) = \frac{1}{s^2}$.

The states are the position and the velocity, but we're only measuring the velocity.

$\rightarrow$ Impossible to estimate the position and therefore this system is ***unobservable***

An LTI system is ***observable*** if and only if we can place the poles of the error system, which can be done only if $\mathcal{O}$ is invertible

LTI system (A, C) is observerable if and only if the observability matrix is full rank

$$\operatorname{rank} \mathcal{O} = n$$

The duality between controller pole placement and observer pole placement means that we can use the same tools

```
K = acker(A, B, pc)
K = place(A, B, pc)
```

```
Lt = acker(A', C', pe)
Lt = place(A', C', pe)
L = Lt'
```

where pc is the list of desired controller poles, and pe is the list of desired estimator poles

Pole Selection

Estimator pole selection is a tradeoff between sensor noise and transient response

- Faster estimator $\rightarrow$ more sensor noise passed to controller
- Slower estimator $\rightarrow$ slower transient response

Rule of thumb: Estimator poles should be faster than the controller poles by about 2-6 times

Example

Design estimator for

$$\dot{x} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Example

Design estimator for

$$\dot{x} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Place the observer poles about $2 - 3\times$ faster than the dominant poles of the system

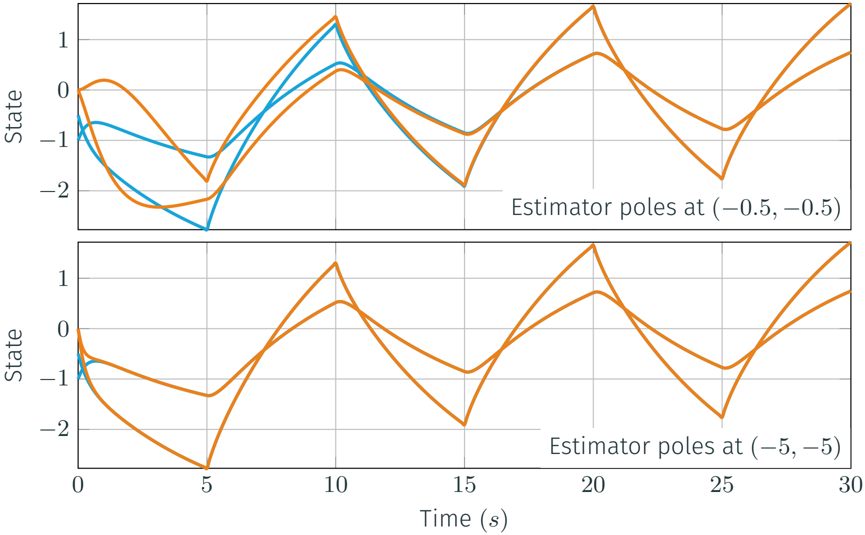
$$\lambda_{\max}(A) = -0.18$$

Place two observer poles at 0.5

```
L = acker(A', C', -[0.5, 0.5])
```

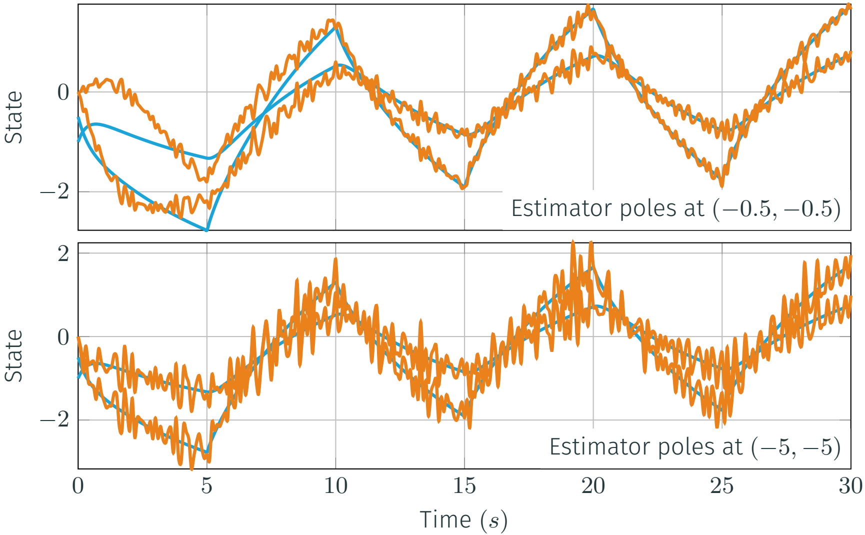
$$L = \begin{bmatrix} -2 \\ 2.5 \end{bmatrix}$$

Example



Blue : True state Orange: Estimated state

Example - Impact of Measurement Noise



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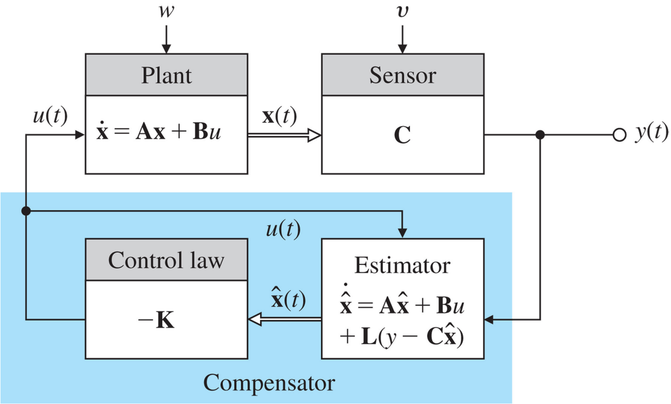
Combining Control and Estimation



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What is the overall system?

Combining Control and Estimation

Full-state feedback controller

$$\begin{aligned}\dot{x} &= Ax - BK\hat{x} \\ &= Ax - BK(x - e)\end{aligned}$$

where $e = x - \hat{x}$

Observer

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} - BK\hat{x} + L(y - C\hat{x}) \\ \dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= Ax - BK\hat{x} - A\hat{x} + BK\hat{x} - L(y - C\hat{x}) \\ &= (A - LC)e\end{aligned}$$

Putting these together gives

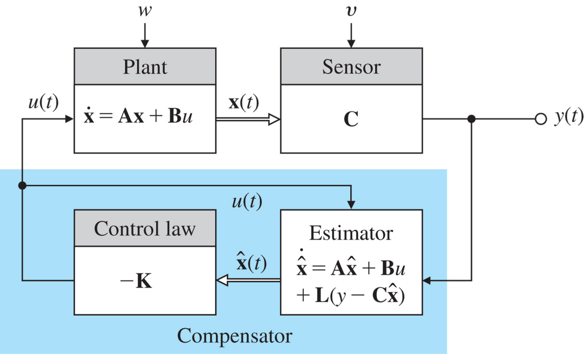
$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}$$

The poles of the closed-loop system are

$$\det(s - A + BK)\det(s - A + LC) = \alpha_c(s)\alpha_e(s) = 0$$

This is called the ***separation principle***.

Controller Transfer Function



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$$\dot{\hat{x}} = (A - LC - BK)\hat{x} + Ly$$
$$u = -K\hat{x}$$

$$\Rightarrow$$

$$K(s) = \frac{U(s)}{Y(s)} = C(sI - A)^{-1}B + D$$
$$= -K(sI - A + LC + BK)^{-1}L$$

Example

Consider the second-order system $G(s) = 1/s^2$

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Design a controller and estimator such that the closed-loop system has an overshoot of no more than 20% and a settling time of 4sec.

Specifications

- $M_p \leq 20\%$ overshoot $\rightarrow \zeta \geq 0.45$
- $T_s \leq 4 \rightarrow \sigma = \zeta\omega_n \geq 1 \rightarrow \omega_n \geq 2.2$

Choose pole locations

$$\alpha_c(s) = s^2 + 2 \cdot 0.45 \cdot 2.2s + 2.2^2$$

Example - Design controller

Place poles by matching characteristic equations (could also use `acker` or `place`)

$$
\begin{aligned}
\alpha_c(s) &= \det\left(sI - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix}\right) \\
&= \det \begin{bmatrix} s + K_1 & K_2 \\ -1 & s \end{bmatrix} \\
&= (s + K_1)s + K_2 = s^2 + K_1 s + K_2 \\
&= s^2 + 2 \cdot 0.45 \cdot 2.2s + 2.2^2
\end{aligned}
$$

choose $K_2 = 5$ and $K_1 = 2$.

Example - Design Observer

Place estimator poles $5\times$ faster than the controller poles

Controller poles have a decay rate of $\sigma = \zeta\omega_n = 1$

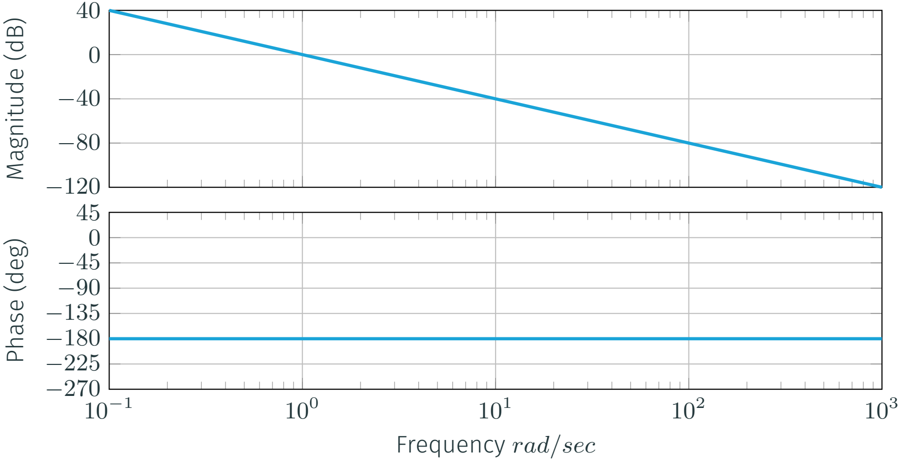
$$\alpha_e(s) = (s+5)^2$$

$$\begin{aligned}
\det(sI - A + LC) &= \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\right) \\
&= \det\begin{bmatrix} s & L_1 \\ -1 & s + L_2 \end{bmatrix} \\
&= s(s + L_2) + L_1 = s^2 + L_2 s + L_1 \\
&= s^2 + 10s + 25
\end{aligned}$$

Choose $L_1 = 25$ and $L_2 = 10$

Example - Regulator

$$K(s) = -K(sI - A + LC + BK)^{-1}L = \frac{100s + 125}{s^2 + 12s + 50}$$



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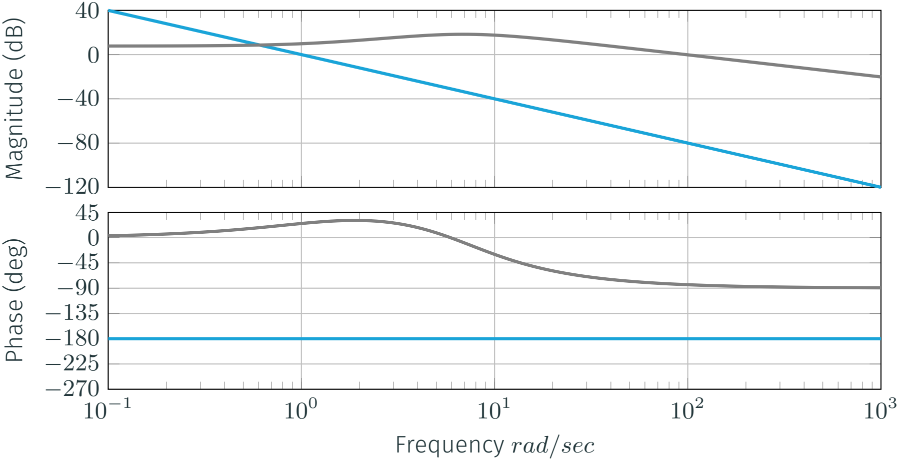

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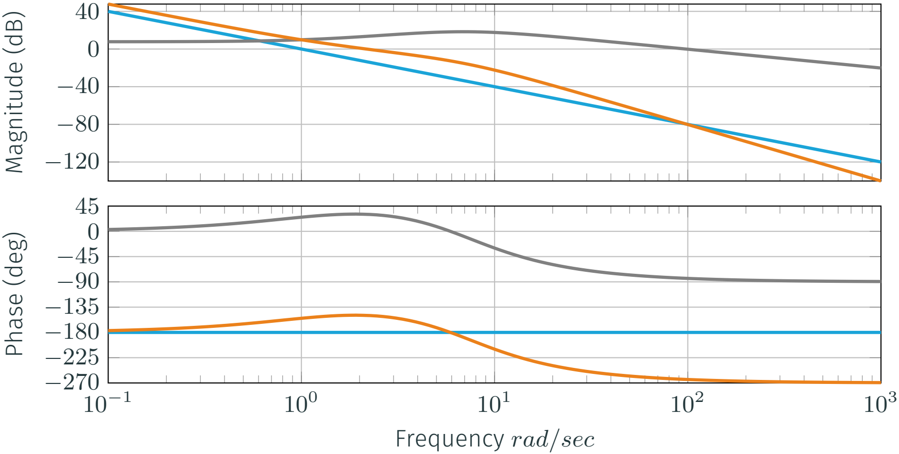


This is a lead compensator ($PM = 30°$, $GM = 12.5dB$)

Reference Input

Where do we Add the Reference?

System and controller dynamic equations

$$
\begin{aligned}
\text{System} \qquad & \dot{x} = Ax + Bu \\
& y = Cx \\
\text{Controller} \qquad & \dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly \\
& u = -K\hat{x}
\end{aligned}
$$

Addition of the reference as a linear input to the controller

$$
\begin{aligned}
\dot{\hat{x}} &= (A - BK - LC)\hat{x} + Ly + Mr \\
u &= -K\hat{x} + \bar{N}r
\end{aligned}
$$

How to choose M and $\bar{N}$?

Note that the reference cannot impact the pole locations. It does change the zeros

Option 1: Autonomous Estimator

Idea:	Estimator is estimating the state of the system, and so should not be impacted by the reference

Select M and $\bar{N}$ so that the state estimation error is independent of r

$$\begin{aligned} \dot{x} - \dot{\hat{x}} &= Ax + B[-K\hat{x} + \bar{N}r] - [(A - BK - LC)\hat{x} + Ly + Mr] \\ \dot{e} &= (A - LC)e + (B\bar{N} - M)r \end{aligned}$$

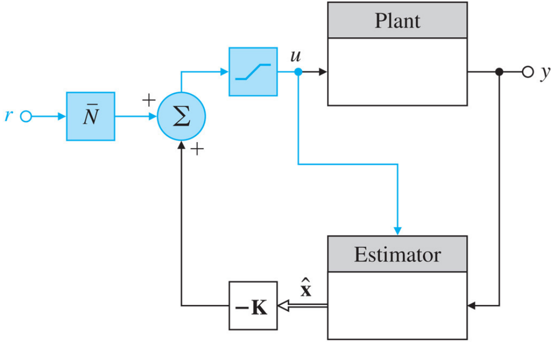
We can see that the reference has no impact if

$$B\bar{N} = M$$

Option 1: Autonomous Estimator

$$\begin{aligned}\dot{\hat{x}} &= (A - BK - LC)\hat{x} + Ly + B\bar{N}r \\ &= (A - LC)\hat{x} + Ly + Bu \\ u &= -K\hat{x} + \bar{N}r\end{aligned}$$

Note that the estimated state $\hat{x}$ does not have the reference as an input



Pro: If the input is saturated, then it can be saturated for the estimator too.

Option 1: Selection of $\bar{N}$

- The reference has no impact on the estimator
- The steady-state estimate equals the true state $\hat{x}_{ss} = x_{ss}$

Choose $\bar{N}$ to ensure zero tracking error in steady-state

$$u = -K\hat{x} + (N_u + KN_x)r = -K\hat{x} + \bar{N}r$$

where N_u and N_x are chosen as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} N_x \\ N_u \end{bmatrix}$$

Note : This is exactly as we saw in the full state-feedback case

Example

Consider the second-order system $G(s) = 1/s^2$

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Design a controller and estimator such that the closed-loop system has an overshoot of no more than 20% and a settling time of 4sec.

We previously computed a controller and observer

$$K = \begin{bmatrix} 2 & 5 \end{bmatrix} \qquad L = \begin{bmatrix} 25 \\ 10 \end{bmatrix}$$

Option 1: Autonomous Estimator

Select the gain $\bar{N}$ so that we have zero steady-state error

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} N_x \\ N_u \end{bmatrix}$$

$$\bar{N} = KN_x + Nu = \begin{bmatrix} 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 = 5$$

Select $M = B\bar{N}$

$$M = \begin{bmatrix} 1 \\ 0 \end{bmatrix} 5 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

The entire system becomes

$$
\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} &= \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{bmatrix} B\bar{N} \\ B\bar{N} \end{bmatrix} r \\
&= \begin{bmatrix} 0 & 0 & -2 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 25 & -2 & -20 \\ 0 & 10 & 1 & -10 \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{bmatrix} 5 \\ 0 \\ 5 \\ 0 \end{bmatrix} r \\
y &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}
\end{aligned}
$$

Poles and Zeros

The poles and zeros of the closed-loop system are

$$\text{Poles} = \begin{pmatrix} -1 \pm 2i & -5 & -5 \end{pmatrix} \qquad \text{Zeros} = \begin{pmatrix} -5 & -5 \end{pmatrix}$$

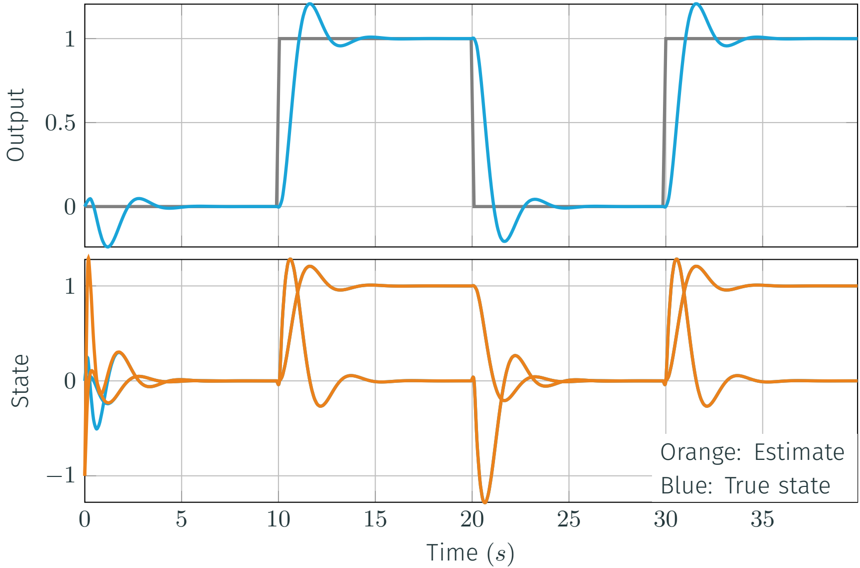
The reference adds zeros to cancel exactly the poles of the estimator

As a result, the system is now uncontrollable.

$$\operatorname{rank} \mathcal{C} = \operatorname{rank} \begin{bmatrix} 5 & -10 & -5 & 60 \\ 0 & 5 & -10 & -5 \\ 5 & -10 & -5 & 60 \\ 0 & 5 & -10 & -5 \end{bmatrix} = 2$$

This makes sense, as we have designed the reference to have no impact on the estimator.

Autonomous Estimator



Option 2: Tracking-error Estimator

Idea: Use only the tracking error $e = r - y$ in the estimator

This form is used when only the error is measured

Choose M and $\bar{N}$ such that the estimator only uses the error $e = r - y$

$$\dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly + Mr$$
$$u = -K\hat{x} + \bar{N}r$$

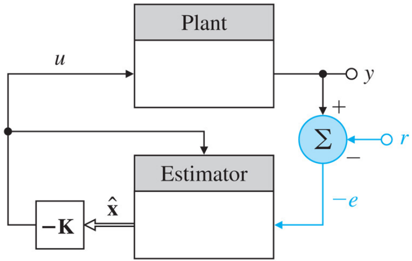
This is satisfied if $\bar{N} = 0$ and $M = -L$

The controller becomes

$$\dot{\hat{x}} = (A - BK - LC)\hat{x} + L(y - r)$$
$$u = -K\hat{x}$$

$$\dot{\hat{x}} = (A - BK - LC)\hat{x} + L(y - r)$$

$$u = -K\hat{x}$$



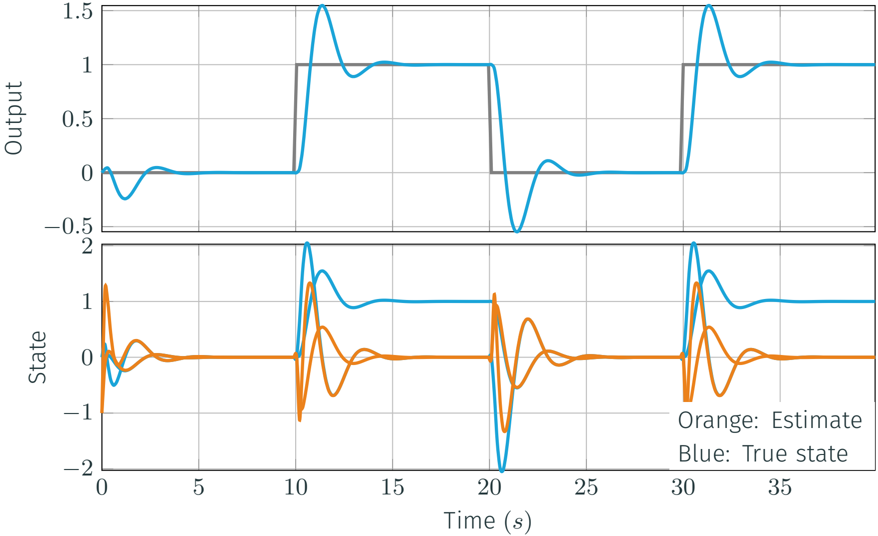
The entire system becomes

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{bmatrix} 0 \\ -L \end{bmatrix} r$$

$$= \begin{bmatrix} 0 & 0 & -2 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 25 & -2 & -20 \\ 0 & 10 & 1 & -10 \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ -25 \\ -10 \end{bmatrix} r$$

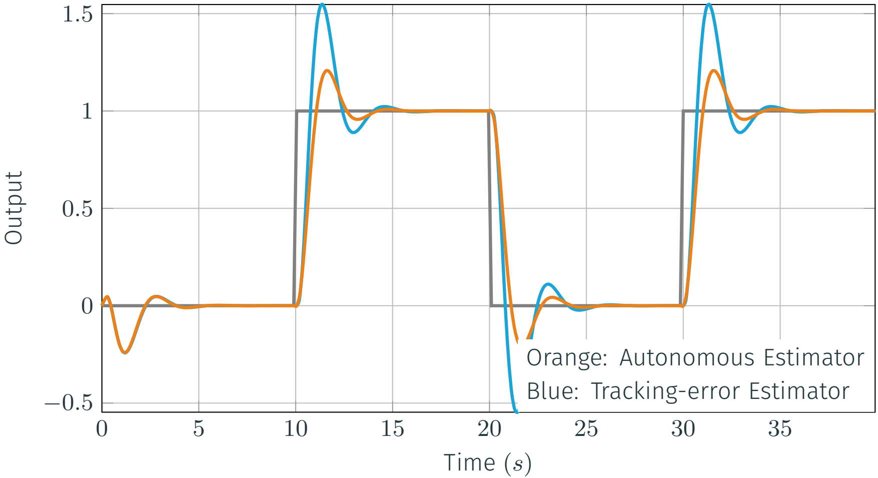
$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$



Note that the estimator is no longer estimating the state.

Comparison of Reference Methods



Tracking-error estimator tends to have larger overshoot.

Integral Control

Integral Control

Problem: No integrator in the control loop. Steady-state offset is likely.
Solution: Add an integrator

Define an artificial state that is the integral of the error

$$x_I(t) = \int_0^t e(\tau)d\tau \qquad \text{where } e(t) = r(t) - y(t) \text{ and } \dot{x}_I(t) = e(t)$$

Define the ***augmented system model*** to include the integral state

$$\begin{pmatrix} \dot{x} \\ \dot{x}_I \end{pmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix}$$

Now design a controller using previous methods and in steady-state we have

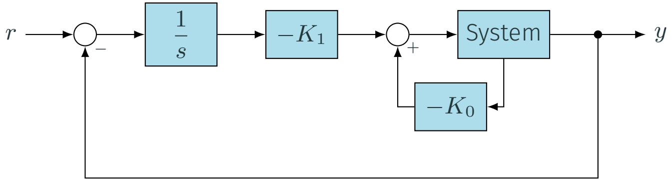
$$\dot{x}_I(t) = e(t) = 0$$

Structure of Integral Controllers

Controller will have the form

$$u = -\begin{bmatrix} K_0 & K_1 \end{bmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix}$$

The resulting control structure is



Note that we've taken a different sign on the feedback loop here compared to the book to keep the standard loop.

Example

$$G(s) = \frac{1}{s+3}$$

Design an offset-free controller with two poles at -5 and an estimator pole at -10.

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State-space model

$$\dot{x} = -3x + u$$

$$y = x$$

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State-space model

$$\dot{x} = -3x + u$$

$$y = x$$

Augment system dynamics

$$\begin{pmatrix} \dot{x} \\ \dot{x}_I \end{pmatrix} = \begin{bmatrix} -3 & 0 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

Example

$$G(s) = \frac{1}{s+3}$$

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Augment system dynamics

$$\begin{pmatrix} \dot{x} \\ \dot{x}_I \end{pmatrix} = \begin{bmatrix} -3 & 0 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

Place poles at $-5, -5$

$$\det sI - A + BK = \det \begin{bmatrix} s+3+K_0 & K_1 \\ 1 & s \end{bmatrix} = (s+3+K_0)s - K_1 = s^2 + 10s + 25$$

Controller is $K = \begin{bmatrix} K_0 & K_1 \end{bmatrix} = \begin{bmatrix} 7 & -25 \end{bmatrix}$

Example

$$G(s) = \frac{1}{s+3}$$

Design an offset-free controller with two poles at -5 and an estimator pole at -10.

State-space model

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$$y = x$$

Place estimator poles

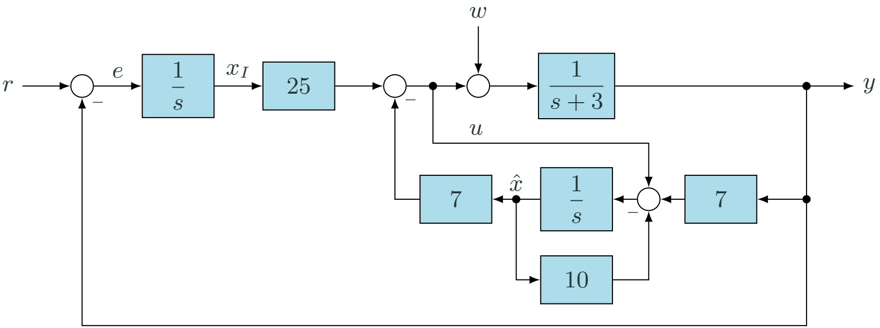
$$\det(sI - A + LC) = s + 3 + L = s + 10$$

Observer gain is $L = 7$

Estimator dynamics

$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu = -10\hat{x} + 7y + u$$

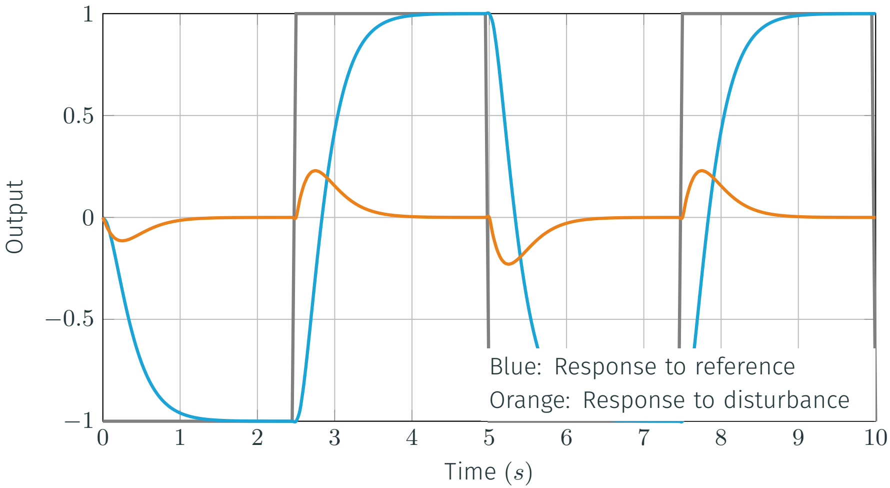
Example - Block Diagram



$$\left.\begin{aligned} \dot{x} &= -3x + u + w \\ u &= -7\hat{x} + 25x_I \\ \dot{x}_I &= r - x \\ \dot{\hat{x}} &= -10\hat{x} + 7x + u \end{aligned}\right\} \quad \begin{pmatrix} \dot{x} \\ \dot{x}_I \\ \dot{\hat{x}} \end{pmatrix} = \begin{bmatrix} -3 & 25 & -7 \\ -1 & 0 & 0 \\ 7 & 25 & -17 \end{bmatrix} \begin{pmatrix} x \\ x_I \\ \hat{x} \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} r + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} w$$

Poles at $(-10, -5, -5)$

Example - Response



We have offset-free tracking and constant disturbance rejection.

Summary - Design Procedure

1 State-Feedback Design

Assume that the ***state is measured***, and design a ***static*** control law $u = Kx$

$$\dot{x} = Ax + BKx$$

Problem : We can't measure x!

2 State Observer

Design a dynamic system to ***estimate the state***

$$\dot{\hat{x}} = L\hat{x} + My + Nu$$

Design L, M and N so that $\hat{x} \sim x$

3. Combine controller and observer to provide a single, dynamic control law.
4. Add reference tracking.

Separation principle tells us that independent design of these elements is ***optimal***.

Pole placement of estimators is ***dual*** to that of controllers.